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FINITE DEFORMATIONS OF INEXTENSIBLE COSSERAT SURFACES

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Abstract - The set of field and constitutive relations, together with the boundary conditions, are derived for finite deformations of inextensible Cosserat surfaces. The theory is applied to the flexure of a rectangular plate into a closed circular cylinder of finite length, and the deformation of a long rectangular plate into a helical strip (barber's pole).

1. INTRODUCTION

THE theory of plates and shells, the middle surface of which can be considered as inextensible, has been the object of several investigations over the last two decades; principal among these are the works by Ashwell [1-3], Mansfield [4], Reissner [5-6] and Johnson and Reissner [7]. The physical significance of the inextensible hypothesis is based on the observation that when the ratio between deflection and thickness of a cantilevered extensible plate increases, its behavior is almost inextensional, and the discrepancies between extensional and inextensional solutions occur in a narrow region concentrated on the edges.

The mechanism by which a boundary layer is created along the edges of an extensible plate in the state of large deformations has been analyzed in detail by Fung and Wittrick [8]. Essentially, if a rectangular strip of width b and thickness h is deformed into a cylindrical strip with longitudinal radius of curvature R, the strip which at first assumes an anticlastic shape, approaches a cylindrical form when the parameter b^2/Rh becomes larger. Disturbances from the cylindrical configuration occur in a narrow boundary layer with a width of the order $\sqrt{(Rh)}$.

When the edge of the plate is also subject to shear force and twisting moment, another type of boundary layer occurs which was originally described by Kelvin and Tait [9], in explaining the physical meaning of Kirchhoff's boundary conditions along a free edge: the width of the corresponding boundary layer is of the order h.

If it is assumed at the outset that the middle surface of the plate is inextensible and that a deformed normal element remains normal to the middle surface, the concept of boundary layer disappears. Instead, it is necessary to introduce three Lagrange multipliers along the edges which allow to satisfy the boundary conditions. This has been done by Reissner [5] for the case of shallow elastic shells, and similar boundary conditions have been found by Ashwell [3] by considering the limit of the extensional problem when the width of the boundary layer vanishes.

One of the Lagrange multipliers introduced in [5] originates from the fact that the edge of an inextensible surface is also inextensible, and an indeterminacy of the boundary

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conditions has to be expected therefrom. Also, the absence of normal shear in [5] and [3] causes the presence of the remaining two indeterminate functions on the edges.

In this paper, we wish to examine the mechanism of large deformations for inextensible Cosserat surfaces, and we follow the general theory of Green *et al.* [10], for elastic surfaces to every point of which a deformable director is attached. Finite deformation solutions dealing essentially with axially and spherically symmetric homogeneous deformations have been obtained recently by Crochet and Naghdi [11].

By inextensible surface we mean that the length of a line element in the surface remains invariant throughout the deformation. However, no constraint is imposed on the director. This allows us to satisfy boundary conditions by introducing only one Lagrange multiplier along the edges, since the director field on the surface can be selected such as to satisfy the moment boundary conditions. This is done in Section 2, where we give also an explicit form of the constitutive relations.

By applying the theory to some specific examples, we find that the effects discussed by Kelvin and Tait [9] and Fung and Wittrick [8] occur in boundary layers along the free edges; normal strain, which is included in the vertical component of the director, is also confined to a region of small width along the boundary.

In Section 3 we consider the problem of a rectangular plate which is rolled into a closed circular cylindrical tube of finite length. In Section 4, we analyze the deformation of a long rectangular strip into a helical strip of constant radius (barber's pole)—an exact solution is obtained which satisfies the boundary conditions along the free edges.

2. AN INEXTENSIBLE ELASTIC COSSERAT SURFACE

Consider a Cosserat surface σ, embedded in a three-dimensional Euclidean space, defined by

$$\mathbf{r} = \mathbf{r}(x^\alpha, t); \qquad \mathbf{d} = \mathbf{d}(x^\alpha, t), \tag{2.1}$$

where $\mathbf{r}$ is the position vector relative to a fixed origin, $\mathbf{d}$ is the director associated with every point of σ, $x^\alpha(\alpha = 1, 2)$ are convected coordinates locating material points on the surface and t is the time. In the reference configuration (say, at time $t = 0$), the position vector is given by $\mathbf{R}(x^\alpha)$, while the director $\mathbf{D}$ has unit length and lies along the normal to the undeformed surface Σ. Also, let $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{w} = \dot{\mathbf{d}}$ be the point and director velocities, respectively.

The base vectors along the x^α-curves on σ are denoted by $\mathbf{a}_\alpha$, and $\mathbf{a}_3$ stands for the unit normal to the surface; their duals on Σ are denoted by $\mathbf{A}_\alpha$ and $\mathbf{A}_3$. Also, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ ($A_{\alpha\beta}$ and $B_{\alpha\beta}$) denote the covariant expression of the first and second fundamental forms, respectively.

Let c be a closed curve lying in σ, and let $\mathbf{v} = v_\alpha \mathbf{a}^\alpha$ be the outward unit normal to c. The curve and director force vectors per unit length of c are denoted by $\mathbf{N}$ and $\mathbf{M}$, respectively, while $\mathbf{F}$ and $\mathbf{L}$ are the surface and director forces per unit area of σ. In component form, we have

$$\mathbf{N} = N^{\alpha i} v_\alpha \mathbf{a}_i, \qquad \mathbf{M} = M^{\alpha i} v_\alpha \mathbf{a}_i \tag{2.2}$$

It was shown in [10] that the field equations have the following form on σ,

$$\begin{aligned} N^{\alpha\beta}|_\alpha - b^\beta_\alpha N^{\alpha 3} + \rho F^\beta &= \rho c^\beta, \\ N^{\alpha 3}|_\alpha + b_{\alpha\beta} N^{\alpha\beta} + \rho F^3 &= \rho c^3, \\ M^{\alpha\beta}|_\alpha - b^\beta_\alpha M^{\alpha 3} + \rho \bar{L}^\beta &= m^\beta, \\ M^{\alpha 3}|_\alpha + b_{\alpha\beta} M^{\alpha\beta} + \rho \bar{L}^3 &= m^3, \end{aligned} \tag{2.3}$$

together with

$$\begin{aligned} \varepsilon_{\beta\alpha}[N^{\alpha\beta} + m^\beta d^\alpha + M^{\gamma\beta}\lambda^\alpha_{\ \gamma}] &= 0 \\ N^{\alpha 3} + (m^3 d^\alpha - m^\alpha d^3) + M^{\gamma 3}\lambda^\alpha_{\ \gamma} - M^{\gamma\alpha}\lambda^3_{\ \gamma} &= 0. \end{aligned} \tag{2.4}$$

In (2.3), $\bar{\mathbf{L}}$ is the difference between the director force $\mathbf{L}$ and the inertia effects due to the motion of the director, $\mathbf{c}$ is the acceleration vector and $\mathbf{m}$ is a vector to be specified by a constitutive relation; in (2.4), the tensor $\lambda_{i\alpha}$ has components given by

$$\lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\alpha\beta} d_3, \qquad \lambda_{3\alpha} = d_{3,\alpha} + b_{\alpha\beta} d^\beta. \tag{2.5}$$

The conservation of mass on the surface is expressed by

$$\rho a^{\frac{1}{2}} = \rho_0 A^{\frac{1}{2}}, \tag{2.6}$$

where ρ, (ρ_0), is the mass density *per unit area* of σ, (Σ) and a, (A), is the determinant of the matrix $a_{\alpha\beta}$, ($A_{\alpha\beta}$). Finally, the equation of balance of energy reduces to

$$\rho r - q^\alpha|_\alpha - \rho T \dot{S} = 0, \tag{2.7}$$

where $\mathbf{q}$ is the heat flux vector, r is the heat supply function per unit time and per unit mass of σ, T is the temperature, and S is the specific entropy per unit mass of the surface.

Constitutive relations for the case of an elastic Cosserat surface have been obtained in [10], with the assumption that the function of free energy A per unit mass of σ has the following form,

$$A = A(T, e_{\alpha\beta}, \lambda_{i\alpha}, d_i, \Lambda_{i\alpha}, D_i), \tag{2.8}$$

where $\Lambda_{i\alpha}$ is the value of $\lambda_{i\alpha}$ in the reference configuration, and

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}. \tag{2.9}$$

Similar constitutive relations were postulated for $N'^{\alpha\beta}$, $M^{i\alpha}$ and m^i, where

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} - m^\alpha d^\beta - M^{\gamma\alpha}\lambda^\beta_{\ \gamma}. \tag{2.10}$$

With the use of Clausius–Duhem inequality, it was found in [10] that

$$\begin{aligned} S &= -\frac{\partial A}{\partial T}, \qquad -q^\alpha T_{,\alpha} \geq 0, \\ m^i &= \rho\frac{\partial A}{\partial d_i}, \qquad M^{\alpha i} = \rho\frac{\partial A}{\partial \lambda_{i\alpha}}, \end{aligned} \tag{2.11}$$

and

$$N'^{\beta\alpha} = \rho\frac{\partial A}{\partial e_{\alpha\beta}}. \tag{2.12}$$

We shall now consider an *inextensible* elastic Cosserat surface, which is constrained so that the length of any element in the surface remains unaltered throughout the motion.†
It follows immediately that for such surfaces,

$$a_{\alpha\beta} = A_{\alpha\beta}, \quad \dot{a}_{\alpha\beta} = 0, \tag{2.13}$$

and from (2.6) and (2.13) we have,

$$\rho = \rho_0 . \tag{2.14}$$

The specific free energy per unit area of the surface is given by

$$A = A(T, \lambda_{i\alpha}, d_i, \Lambda_{i\alpha}). \tag{2.15}$$

Through a procedure similar to that used in [10], it is easy to show that the constitutive relations (2.11) remain unchanged for the case of an inextensible surface. However, $N^{\alpha\beta}$ formerly given by (2.12) remains undetermined, and from (2.10) we obtain

$$N^{\alpha\beta} = T^{\alpha\beta} + m^\alpha d^\beta + M^{i\alpha}\lambda_i^{\ \beta} , \tag{2.16}$$

where $T^{\alpha\beta}$ in an arbitrary symmetric tensor which should be determined through the equations of motion.

In order to obtain the form of the boundary conditions, let us consider an inextensible Cosserat surface σ which is acted upon by a surface force field $\mathbf{F}(\mathbf{x})$, and a director force field $\mathbf{L}(\mathbf{x})$; the boundary curve c^* of σ supports a piecewise continuous curve force vector $\mathbf{N}^*(c)$ and a continuous director force $\mathbf{M}^*(c)$, both per unit length of c^*, and the heat flux along the boundary has the value $h^*(c)$. The equation of balance of energy postulated in [10] must be satisfied for the surface σ surrounded by c^*, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\sigma [\tfrac{1}{2}\rho\mathbf{v}\,.\,\mathbf{v} + \rho(A+TS)]\,\mathrm{d}\sigma - \int_\sigma [\rho r + \rho\mathbf{F}\,.\,\mathbf{v} + \rho\bar{\mathbf{L}}\,.\,\mathbf{w}]\,\mathrm{d}\sigma$$
$$= \int_{c^*} [\mathbf{N}^*\,.\,\mathbf{v} + \mathbf{M}^*\,.\,\mathbf{w} - h^*]\,\mathrm{d}c + \sum_{i=1}^{n} \mathbf{P}^{(i)}\,.\,\mathbf{v}^{(i)}, \tag{2.17}$$

where n point loads act at distinct corner points $\mathbf{x}^{(i)}$ along the boundary c^*. We shall also consider that all necessary conditions of continuity and differentiability are satisfied throughout σ.

With the use of the field equations (2.3), (2.4), (2.7), and the constitutive relations (2.11), (2.15) and (2.16), it is possible to transform the left-hand side of (2.17) as follows,‡

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\sigma [\tfrac{1}{2}\rho\mathbf{v}\,.\,\mathbf{v} + \rho(A+TS)]\,\mathrm{d}\sigma - \int_\sigma [\rho r + \rho\mathbf{F}\,.\,\mathbf{v} + \rho\bar{\mathbf{L}}\,.\,\mathbf{w}]\,\mathrm{d}\sigma$$
$$= \int_\sigma [\rho(\dot{\mathbf{v}} - \mathbf{F})\,.\,\mathbf{v} + \rho(T\dot{S} - r) + m^i\dot{d}_i + M^{\alpha i}\dot{\lambda}_{i\alpha} - \rho\bar{\mathbf{L}}\,.\,\mathbf{w}]\,\mathrm{d}\sigma$$
$$= \int_\sigma \mathbf{T}^{\alpha}{}_{|\alpha}\,.\,\mathbf{v}\,\mathrm{d}\sigma - \int_{c^*} [(\mathbf{N}^* - \mathbf{T}^\alpha\nu_\alpha)\,.\,\mathbf{v} + \mathbf{M}^\alpha\nu_\alpha\,.\,\mathbf{w} - q^\alpha\nu_\alpha]\,\mathrm{d}c\,. \tag{2.18}$$

† We make a distinction between the notions of "inextensible Cosserat surface" and "inextensional deformation of a Cosserat surface". The latter refers to a special type of deformation of a general surface while the former indicates a special property of the surface itself.

‡ In deriving (2.18), we have not written explicitely the related kinematical steps. The derivation amounts to following in inverse order the argument in [10] for obtaining the field relations from the equation of balance of energy.

where we have introduced the notation,

$$\mathbf{N}^\alpha = N^{\alpha i}\mathbf{a}_i, \qquad \mathbf{T}^\alpha = T^{\alpha\beta}\mathbf{a}_\beta, \qquad \mathbf{M}^\alpha = M^{\alpha i}\mathbf{a}_i. \tag{2.19}$$

Thus, by (2.17) and (2.18), we obtain the identity,

$$\int_\sigma \mathbf{T}^\alpha|_\alpha \cdot \mathbf{v}\,\mathrm{d}\sigma + \int_{c^*} \{[(\mathbf{N}^\alpha - \mathbf{T}^\alpha)\nu_\alpha - \mathbf{N}^*]\cdot\mathbf{v} + (\mathbf{M}^\alpha\nu_\alpha - \mathbf{M}^*)\cdot\mathbf{w} - (q^\alpha\nu_\alpha - h^*)\}\,\mathrm{d}c$$
$$- \sum_{i=1}^{n} \mathbf{P}^{(i)}\cdot\mathbf{v}^{(i)} = 0. \tag{2.20}$$

Consider at time t a given deformation of the Cosserat surface, and let the state of stress be specified by the vectors $\mathbf{T}^\alpha$, $\mathbf{N}^\alpha$, $\mathbf{M}^\alpha$ on σ, and $\mathbf{N}^*$, $\mathbf{M}^*$ on the boundary c^*. Let $\mathbf{v}$, $\mathbf{w}$ be the associated point and director velocity fields at time t. From the form of the constitutive relations (2.11) and (2.16), we observe that another velocity field, $\bar{\mathbf{v}}$, $\bar{\mathbf{w}}$ say, may be associated to the stress field specified above, with the same state of deformation. We may conclude that (2.20) must be satisfied for all point and director velocity fields which are compatible with the constraints.

Let $x^\alpha = x^\alpha(c)$ be the parametric equations of the boundary c^*: the length of a line element along c^* remains constant in time, and we have,

$$a_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = a_{\alpha\beta}\frac{\partial x^\alpha}{\partial c}\frac{\partial x^\beta}{\partial c}\mathrm{d}c^2$$
$$= a_{\alpha\beta}t^\alpha(c)t^\beta(c)\,\mathrm{d}c^2 = \text{const.}, \tag{2.21}$$

where t^α are the components of the tangent vector $\mathbf{t}^*$ to c^*. From (2.21) we conclude that along c^*,

$$\dot{a}_{\alpha\beta}t^\alpha(c)t^\beta(c) = 0. \tag{2.22}$$

Also, $(2.13)_2$ must be satisfied throughout σ.

Let $S^{\alpha\beta}(\mathbf{x})$ be an arbitrary symmetric tensor field which is continuous and differentiable over σ, and let $\Lambda(c)$ be an arbitrary scalar which is piecewise continuous and differentiable over the boundary c^*, with possible discontinuities at points $\mathbf{x}^{(i)}$, $i = 1, \ldots, n$. By considering $S^{\alpha\beta}(\mathbf{x})$ and $\Lambda(c)$ as Lagrange multipliers, and taking into account the constraints $(2.13)_2$ and (2.22), we shall add to the identity (2.20) the following expression,

$$\frac{1}{2}\left\{\int_\sigma S^{\beta\alpha}\dot{a}_{\alpha\beta}d\sigma + \int_{c^*}\Lambda(c)\dot{a}_{\alpha\beta}t^\alpha t^\beta\,\mathrm{d}c\right\}$$
$$= \int_\sigma \mathbf{S}^\beta\cdot\mathbf{v}_{,\beta}\,\mathrm{d}\sigma + \int_{c^*}\Lambda(c)\mathbf{t}(c)\cdot\frac{\partial\mathbf{v}}{\partial c}\,\mathrm{d}c$$
$$= \int_\sigma -\mathbf{S}^\beta|_\beta\cdot\mathbf{v}\,\mathrm{d}\sigma + \int_{c^*}\left\{\mathbf{S}^\beta\nu_\beta - \frac{\partial}{\partial c}[\Lambda(c)\mathbf{t}(c)]\right\}\cdot\mathbf{v}\,\mathrm{d}c + \sum_{i=1}^{n}[\Lambda(c)\mathbf{t}(c)]_{i-}^{i+}\cdot\mathbf{v}^{(i)} \tag{2.23}$$

where

$$\mathbf{S}^\beta = S^{\beta\alpha}\mathbf{a}_\alpha. \tag{2.24}$$

By adding (2.23) to (2.20) we obtain

$$\int_{c^*} \left\{ \left[(\mathbf{N}^\alpha + \mathbf{S}^\alpha - \mathbf{T}^\alpha)\nu_\alpha - \mathbf{N}^* - \frac{\partial}{\partial c}(\Lambda(c)\mathbf{t}(c)) \right] . \mathbf{v} + (\mathbf{M}^\alpha \nu_\alpha - \mathbf{M}^*) . \mathbf{w} + (q^\alpha \nu_\alpha - h^*) \right\} \mathrm{d}c$$
$$+ \int_\sigma (\mathbf{T}^\alpha|_\alpha - \mathbf{S}^\beta|_\beta) . \mathbf{v}\, \mathrm{d}\sigma + \sum_{i=1}^{n} \{[\Lambda(c)\mathbf{t}(c)]_i^{+} - \mathbf{P}^{(i)}\} . \mathbf{v}^{(i)} = 0. \qquad (2.25)$$

Equation (2.25) must be identically satisfied for arbitrary (continuous) point and director velocity fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$ respectively, over σ and c^*. From the surface integral in (2.25), it follows that $\mathbf{T}^\alpha$ and $\mathbf{S}^\alpha$ differ only through a vector $\mathbf{U}^\alpha$ such that

$$\mathbf{U}^\alpha|_\alpha = 0. \qquad (2.26)$$

However, $\mathbf{U}^\alpha$ may be set to vanish identically without loss of generality, since it is contained in the general solution of the homogeneous part of $(2.3)_1$. In view of the continuity requirements on the variables appearing in the line integral on the left-hand side of (2.25), we conclude that on smooth elements of the boundary c^* we have†

$$\mathbf{N}^\alpha \nu_\alpha = \mathbf{N}^* + \frac{\partial}{\partial c}(\Lambda(c)\mathbf{t}(c)),$$
$$\mathbf{M}^\alpha \nu_\alpha = \mathbf{M}^*, \qquad (2.27)$$

and at the corners we find

$$[\Lambda(c)\mathbf{t}(c)]_i^{+} = \mathbf{P}^{(i)}. \qquad (2.28)$$

Then it follows easily from (2.25), (2.27) and (2.28) that

$$q^\alpha \nu_\alpha = h^*. \qquad (2.29)$$

We now return to the constitutive relation (2.15) for the specific free energy of the surface, and consider a Cosserat *plate* which is isotropic in the reference configuration. For a plate, $\Lambda_{i\alpha}$ vanishes identically, and A may be expressed in terms of fifteen invariants (obtained from those given in [10] by setting $e_{\alpha\beta} = 0$). However, in the rest of this paper, we shall assume that for isothermal processes at temperature T_0 the free energy A can be written as a quadratic function of the variables $(d_3 - 1)$, d_α and $\lambda_{i\alpha}$.‡ If the Cosserat plate imitates the symmetry of a three-dimensional plate which is transversely isotropic with respect to normals to the plate, the free energy has the form

$$\rho_0 A = \tfrac{1}{2}\alpha_3 d^\alpha d_\alpha + \tfrac{1}{2}\alpha_4 (d_3 - 1)^2 + \tfrac{1}{2}\alpha_8 \lambda_{3\alpha}\lambda_3^{\ \alpha}$$
$$+ \tfrac{1}{2}[\alpha_5 a^{\alpha\beta} a^{\gamma\delta} + \alpha_6 a^{\alpha\gamma} a^{\beta\delta} + \alpha_7 a^{\alpha\delta} a^{\beta\gamma}] \lambda_{\alpha\beta}\lambda_{\gamma\delta} \qquad (2.30)$$

and constitutive relations $(2.11)_{2,3}$ assume the simple form

$$m_\alpha = \alpha_3 d_\alpha, \qquad m_3 = \alpha_4 (d_3 - 1),$$
$$M_{\alpha\beta} = \alpha_5 \lambda^\gamma_{\ \gamma} a_{\alpha\beta} + \alpha_6 \lambda_{\beta\alpha} + \alpha_7 \lambda_{\alpha\beta}, \qquad (2.31)$$
$$M_{\alpha 3} = \alpha_8 \lambda_{3\alpha}.$$

† Let us recall that the boundary conditions for an extensible Cosserat surface are given by $\mathbf{N}^\alpha \nu_\alpha = \mathbf{N}^*$, $\mathbf{M}^\alpha \nu_\alpha = \mathbf{M}^*$ on c^*

‡ No linear terms will be included since we consider a plate which is free of stress in the reference configuration

In a previous paper on special solutions for Cosserat surfaces, Crochet and Naghdi [11] have used a general form for the function of free energy, and have shown that some special solutions can be obtained without specialized assumptions. However, in view of the purpose of the present paper, we shall assume for simplicity that the constitutive relations are given by (2.31). It may happen that the form (2.30) for the free energy A is exact for a particular type of Cosserat plate. Alternatively, if A given by (2.15) has a polynomial expansion, the form (2.30) may be regarded as an approximation for motions in which the magnitudes of the vector $(\mathbf{d}-\mathbf{1})$ and tensor $\{\lambda_{i\alpha}\}$ (in a suitable non-dimensional form) are small compared to 1.

3. BENDING OF A RECTANGULAR PLATE INTO A CLOSED CIRCULAR CYLINDER

Let (x, y, z) stand for the coordinates of a point in a rectangular Cartesian system, and consider an inextensible rectangular Cosserat plate which, in its reference configuration, lies in the plane $z = 0$; its area Σ is defined by

$$\Sigma: 0 \leq x \leq L, \qquad -a \leq y \leq a, \qquad z = 0. \tag{3.1}$$

Let (r, θ, ζ) denote the coordinates of a point in a cylindrical polar coordinates system [the origin of which does not necessarily coincide with that of the (x, y, z) system], and let $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\zeta$ be unit vectors tangent to coordinate lines at a generic point in space. The surface σ is deformed into a closed circular cylinder of radius $R = L/2\pi$, and axial length $2a$, such that the position of a point (x, y) of Σ is given in the deformed configuration by,

$$r = \frac{L}{2\pi}, \qquad \theta = \frac{2\pi}{L}x, \qquad \zeta = y. \tag{3.2}$$

The description of the deformed Cosserat surface is achieved by requiring, from axial symmetry considerations, that the deformed director at a given point of σ has no component along $\mathbf{e}_\theta$, and that its value does not depend on θ. Moreover, we shall assume that all partial derivatives with respect to θ- or x-vanish identically. Finally, we shall assume that the deformed surface is static, that σ is free of surface and director forces, and its edges c^* are also free of applied forces and moments: thus,

$$\begin{aligned} \mathbf{c} = \mathbf{F} = \bar{\mathbf{L}} = \mathbf{0}, &\qquad \text{on } \sigma \\ \mathbf{N}^* = \mathbf{M}^* = 0, &\qquad \text{on } c^*. \end{aligned} \tag{3.3}$$

We shall select the convected coordinates (x^1, x^2) to coincide with the coordinates (x, y) in the reference configuration, and

$$x^1 = x, \qquad x^2 = y, \quad \text{on } \Sigma. \tag{3.4}$$

The base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ coincide with $\mathbf{e}_\theta, \mathbf{e}_\zeta, \mathbf{e}_r$, respectively, and we have

$$\begin{aligned} a_{\alpha\beta} = \delta_{\alpha\beta}, &\qquad a^{\alpha\beta} = \delta^{\alpha\beta}, \\ b_{11} = -\frac{1}{R}, &\qquad b_{12} = b_{21} = b_{22} = 0, \end{aligned} \tag{3.5}$$

which guarantees that the deformation is inextensional, and shows that no distinction has to be made between covariant and contravariant tensors. Moreover, all Christofel symbols vanish identically, and all covariant derivatives reduce to the common derivative. The director components in the deformed configuration are given by,

$$d_1 = 0, \qquad d_2 = d_2(y), \qquad d_3 = d_3(y). \tag{3.6}$$

With the use of (2.5), (3.5), (3.6) and (2.31), we find that

$$\begin{aligned} & m_1 = 0, \qquad m_2 = \alpha_3 d_2, \qquad m_3 = \alpha_4(d_3 - 1), \\ & M_{11} = (\alpha_5 + \alpha_6 + \alpha_7)\frac{1}{R}d_3 - \alpha_5\frac{\partial d_2}{\partial y}, \\ & M_{22} = \alpha_5\frac{d_3}{R} + (\alpha_5 + \alpha_6 + \alpha_7)\frac{\partial d_2}{\partial y}, \\ & M_{12} = M_{21} = M_{13} = 0, \qquad M_{23} = \alpha_8\frac{\partial d_3}{\partial y}, \end{aligned} \tag{3.7}$$

and from $(2.4)_2$, it follows that

$$N_{13} = 0, \tag{3.8}$$

while the value of N_{23} is provided by $(2.4)_2$ in terms of quantities given in (3.7). From symmetry considerations, we have†

$$N_{12} = N_{21} = 0, \qquad T_{12} = T_{21} = 0, \tag{3.9}$$

where we have used (2.16) and (3.7).

By taking into account (3.3) and (3.5), we may solve the equations of equilibrium $(2.3)_1$ and obtain

$$N_{22} = C, \qquad N_{11} = R\frac{\partial N_{23}}{\partial y}, \tag{3.10}$$

where C is a constant. In view of (3.3), (3.5), (3.7), the equations of equilibrium $(2.3)_2$ for director forces become

$$\begin{aligned} (\alpha_5 + \alpha_6 + \alpha_7)\frac{\partial^2 d_2}{\partial y^2} - \alpha_3 d_2 + \alpha_5\frac{1}{R}\frac{\partial d_3}{\partial y} = 0, \\ \alpha_8\frac{\partial^2 d_3}{\partial y^2} - \left[\alpha_4 + \frac{1}{R^2}(\alpha_5 + \alpha_6 + \alpha_7)\right]d_3 - \alpha_5\frac{1}{R}\frac{\partial d_2}{\partial y} + \alpha_4 = 0. \end{aligned} \tag{3.11}$$

From the symmetry of the problem with respect to the plane $z = 0$, the general solution of (3.11) is easily found to be

$$\begin{aligned} d_2 &= \Delta_2 \operatorname{sh} \mu_2 y + \Delta_3 \operatorname{sh} \mu_3 y, \\ d_3 &= d_3^0 + f(\mu_2)\Delta_2 \operatorname{ch} \mu_2 y + f(\mu_3)\Delta_3 \operatorname{ch} \mu_3 y, \end{aligned} \tag{3.12}$$

† Equations (3.9) can also be obtained by solving field equations and applying boundary conditions. However, it is found easier to accept (3.9) at the outset.

where

$$d_3^0 = \frac{\alpha_4 R^2}{\alpha_4 R^2 + (\alpha_5 + \alpha_6 + \alpha_7)},$$
$$f(\mu) = \frac{-R}{\alpha_5 \mu}[(\alpha_5 + \alpha_6 + \alpha_7)\mu^2 - \alpha_3], \tag{3.13}$$

and $(\mu_2)^2$, $(\mu_3)^2$ are the roots of

$$[(\alpha_5 + \alpha_6 + \alpha_7)\mu^2 - \alpha_3]\left[\alpha_8 \mu^2 - \frac{1}{R^2}(\alpha_4 R^2 + \alpha_5 + \alpha_6 + \alpha_7)\right] + \left(\frac{\alpha_5}{R}\right)^2 \mu^2 = 0. \tag{3.14}$$

The values of the constants C, Δ_2 and Δ_3 will be obtained from the boundary conditions. Along the boundary curve $y = a$, we have

$$\mathbf{v} = \mathbf{a}_2, \qquad c = x, \qquad \mathbf{t}(x) = \mathbf{a}_1, \tag{3.15}$$

and with the use of (3.3), (3.7) and (3.9), conditions (2.27) can be written as follows

$$\mathbf{N}^2 = N_{22}\mathbf{a}_2 + N_{23}\mathbf{a}_3$$
$$= \frac{\partial}{\partial x}(\Lambda(x)\mathbf{a}_1) = \frac{\partial \Lambda(x)}{\partial x}\mathbf{a}_1 - \frac{1}{R}\Lambda(x)\mathbf{a}_3, \tag{3.16}$$
$$\mathbf{M}^2 = M_{22}\mathbf{a}_2 + M_{23}\mathbf{a}_3 = 0.$$

From (3.16) we obtain the conditions

$$\Lambda(x) = \Lambda = \text{const.} = -RN_{23}(a),$$
$$N_{22}(a) = C = 0, \tag{3.17}$$
$$M_{22}(a) = M_{23}(a) = 0.$$

The constants Δ_2 and Δ_3 can be determined with the use of (3.7) and the last two equations of (3.17),

$$\mu_2 f(\mu_2)\,\mathrm{sh}\,\mu_2 a\,\Delta_2 + \mu_3 f(\mu_3)\,\mathrm{sh}\,\mu_3 a\,\Delta_3 = 0,$$
$$\left[f(\mu_2) + R\frac{\alpha_5 + \alpha_6 + \alpha_7}{\alpha_5}\mu_2\right]\mathrm{ch}\,\mu_2 a\,\Delta_2 + \left[f(\mu_3) + R\frac{\alpha_5 + \alpha_6 + \alpha_7}{\alpha_5}\mu_3\right]\mathrm{ch}\,\mu_3 a\,\Delta_3 = -d_{3_0}. \tag{3.18}$$

By allowing the presence of directors on the elastic Cosserat surface, we have thus shown that the boundary conditions on moments and tangential resultant forces can be fully satisfied along the free edges on a closed circular cylinder. However the normal shear component of the resultant force along the free edges does not vanish and is given by $(2.4)_2$: this result should be expected, since the circular edge is inextensible, and is able to support an indeterminate amount of radial force without enduring deformations. The form of equations (3.12) suggests that the deformation of the directors might be concentrated on a boundary layer along the free edges; however, this can be verified only after specific values have been indicated for the coefficients α_i. We will do this in the next Section, where a more general configuration than the special deformation described above will be discussed.

If the present problem were solved by using the classical theory of inextensible plates, there would be no way of satisfying the boundary condition

$$M_{22}(\pm a) = 0 \tag{3.19}$$

along the edges. The component M_{22}, which gives rise to the anticlastic curvature when the plate is extensible, does now vanish on the boundary and the director separates from the normal along the edges. The physical meaning of this is evident if one compares the inextensible Cosserat surface to an inextensible sheet being sandwiched between two layers of elastic material. If such a rectangular plate is rolled into a closed cylinder, the middle sheet becomes perfectly cylindrical, while the elastic layers are deformed near the edges.

4. BENDING OF A PLATE INTO A HELICAL STRIP

We consider now an inextensible Cosserat plate of infinite length and width $2a$ which, in its reference configuration, lies in the plane $z = 0$ of a rectangular Cartesian system (x, y, z), and its area Σ is defined by

$$\Sigma: -\infty < x < \infty, \qquad -a \leq y \leq -a, \qquad z = 0. \tag{4.1}$$

The plate is deformed into a *helical strip* of radius R and angle of pitch α (Fig. 1) such that a material line which is initially parallel to the x-axis becomes a helicoidal curve of slope α on a circular cylinder of radius R. If (r, θ, ζ) denote the coordinates of a point in a cylindrical polar coordinate system, the deformation of the Cosserat plate is fully characterized by

$$\zeta = \sin\alpha\, x + \cos\alpha\, y, \qquad \theta = \frac{1}{R}\cos\alpha\, x - \frac{1}{R}\sin\alpha\, y,$$
$$r = R, \qquad \mathbf{d} = \mathbf{d}(y), \tag{4.2}$$

where a point with initial coordinates $(x, y, 0)$ occupies a position (r, θ, ζ) in the deformed configuration.

We shall assume that the deformed surface is static, and free of surface and director forces. In addition, the edges $y = \pm a$ are free of applied forces and moments: thus

$$\mathbf{c} = \mathbf{F} = \overline{\mathbf{L}} = 0 \quad \text{on } \sigma,$$
$$\mathbf{N}^* = \mathbf{M}^* = 0, \qquad y = \pm a. \tag{4.3}$$

The convected coordinates are selected such as to coincide with the coordinates (x, y) in the reference configuration. The base vectors $\mathbf{a}_i$ are easily determined with the use of

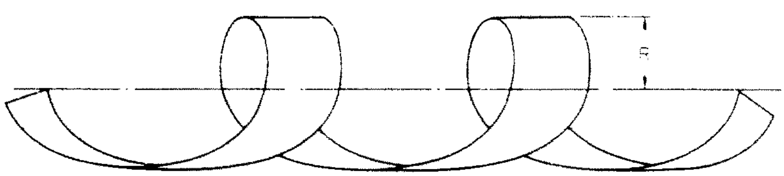

FIG. 1. A helical strip.

(4.2) in terms of the unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ defined in Section 3, and we have,

$$\mathbf{a}_1 = \cos\alpha\,\mathbf{e}_\theta + \sin\alpha\,\mathbf{e}_z, \qquad \mathbf{a}_2 = -\sin\alpha\,\mathbf{e}_\theta + \cos\alpha\,\mathbf{e}_z, \qquad \mathbf{a}_3 = \mathbf{e}_r. \tag{4.4}$$

It follows also from (4.4) that

$$a^{\alpha\beta} = a_{\alpha\beta} = A^{\alpha\beta} = A_{\alpha\beta} = \delta_{\alpha\beta},$$
$$b_{11} = -\frac{\cos^2\alpha}{R}, \qquad b_{22} = -\frac{\sin^2\alpha}{R}, \qquad b_{12} = b_{21} = \frac{\sin\alpha\cos\alpha}{R}; \tag{4.5}$$

(4.5) shows that the deformation is inextensional. Again, covariant derivatives reduce to the common derivative, and the distinction between contravariant and covariant tensors is immaterial.

From $(4.2)_4$, the director $\mathbf{d}$ is written as follows,

$$\mathbf{d} = d_1(y)\mathbf{a}_1 + d_2(y)\mathbf{a}_2 + d_3(y)\mathbf{a}_3. \tag{4.6}$$

With the help of (4.5), and by assuming from axial symmetry that curve forces do not depend on the coordinate x, we find that the equilibrium equations $(2.3)_{1-2}$ for curve forces become

$$\frac{\partial N_{21}}{\partial y} + \frac{\cos\alpha}{R}(\cos\alpha\, N_{13} - \sin\alpha\, N_{23}) = 0,$$
$$\frac{\partial N_{22}}{\partial y} - \frac{\sin\alpha}{R}(\cos\alpha\, N_{13} - \sin\alpha\, N_{23}) = 0, \tag{4.7}$$
$$\frac{\partial N_{23}}{\partial y} - \frac{\cos^2\alpha}{R}N_{11} + \frac{\sin\alpha\cos\alpha}{R}(N_{12}+N_{21}) - \frac{\sin^2\alpha}{R}N_{22} = 0.$$

The components N_{13} and N_{23} of the curve force vector can be expressed as a function of the director components with the use of $(2.4)_2$, (2.5), (4.5), (4.6) and (2.31); the symmetric part of $N_{\alpha\beta}$ can also be expressed as a function of N_{21} together with the director components through the use of the same equations since from $(2.4)_1$ we find,

$$N_{12} + N_{21} = 2N_{21} + m_1 d_2 - m_2 d_1 + M_{\gamma 1}\lambda_{2\gamma} - M_{\gamma 2}\lambda_{1\gamma}. \tag{4.8}$$

The solution of (4.7) in terms of the director components is then easily obtained in the following form

$$N_{21} = -\frac{\cos\alpha}{R}\int_a^y [\cos\alpha\, N_{13}(\tau) - \sin\alpha\, N_{23}(\tau)]\,d\tau + B,$$
$$N_{22} = -tg\alpha\, N_{21} + C,$$
$$N_{11} = \frac{R}{\cos^2\alpha}\frac{\partial N_{23}}{\partial y} + tg\alpha\,(N_{12}+N_{21}) - tg^2\alpha\, N_{22},$$

where B and C are integration constants. In order to calculate the boundary conditions for curve forces, we note that on the boundary $y = a$ we have,

$$\mathbf{v} = (0, 1), \qquad c = x,$$
$$\mathbf{t} = \mathbf{a}_1, \qquad \frac{\partial \mathbf{t}}{\partial c} = \frac{\partial \mathbf{a}_1}{\partial x} = -\frac{\cos^2\alpha}{R}\mathbf{a}_3. \tag{4.10}$$

and it follows from (2.27) and (4.3) that on the boundary $y = a$

$$\begin{aligned} \mathbf{N}^2 &= N_{21}(a)\mathbf{a}_1 + N_{22}(a)\mathbf{a}_2 + N_{23}(a)\mathbf{a}_3 \\ &= \frac{\partial \Lambda(x)}{\partial x}\mathbf{a}_1 - \frac{\cos^2\alpha}{R}\Lambda(x)\mathbf{a}_3, \end{aligned} \tag{4.11}$$

for all $-\infty \leq x \leq \infty$. Thus,

$$\Lambda(x) = -\frac{R}{\cos^2\alpha} N_{23}(a), \tag{4.12}$$

and since $N_{23}(a)$ does not depend on x, we have,

$$\frac{\partial \Lambda(x)}{\partial x} = 0, \qquad N_{21}(a) = 0, \qquad N_{22}(a) = 0, \tag{4.13}$$

which, together with (4.9), shows that

$$B = C = 0. \tag{4.14}$$

In view of the symmetry of the problem with respect to the x axis, the boundary conditions for curve forces will be automatically satisfied for $y = -a$.

With the use of (2.5), (2.31), (4.3), (4.5) and (4.6), the equations of equilibrium for director forces $(2.3)_{3,4}$ may be written in terms of the director components, and read as follows

$$\begin{aligned} &\alpha_6 \frac{\partial^2 d_1}{\partial y^2} - \left(\alpha_3 + \alpha_8 \frac{\cos^2\alpha}{R^2}\right) d_1 + \alpha_8 \frac{\sin\alpha\cos\alpha}{R^2} d_2 \\ &\qquad - (\alpha_6 + \alpha_7 + \alpha_8)\frac{\sin\alpha\cos\alpha}{R}\frac{\partial d_3}{\partial y} = 0, \\ &\alpha_8 \frac{\sin\alpha\cos\alpha}{R^2} d_1 + (\alpha_5 + \alpha_6 + \alpha_7)\frac{\partial^2 d_2}{\partial y^2} - \left(\alpha_3 - \alpha_8\frac{\sin^2\alpha}{R^2}\right) d_2 \\ &\qquad + \frac{1}{R}[\alpha_5 + (\alpha_6 + \alpha_7 + \alpha_8)\sin^2\alpha]\frac{\partial d_3}{\partial y} = 0, \\ &(\alpha_6 + \alpha_7 + \alpha_8)\frac{\sin\alpha\cos\alpha}{R}\frac{\partial d_1}{\partial y} - \frac{1}{R}[\alpha_5 + (\alpha_6 + \alpha_7 + \alpha_8)\sin^2\alpha]\frac{\partial d_2}{\partial y} \\ &\qquad - \alpha_8\frac{\partial^2 d_3}{\partial y^2} - \left[\alpha_4 + \frac{1}{R^2}(\alpha_5 + \alpha_6 + \alpha_7)\right] d_3 = -\alpha_4. \end{aligned} \tag{4.15}$$

The general solution of the system (4.15) can be easily obtained in terms of exponentials. However, before proceeding further, we wish to write the coefficients α_i of the free energy polynomial in nondimensional form.

In two recent publications [12, 13], Green and Naghdi have shown the correspondence between the linear theory of an elastic Cosserat surface and problems concerned with elastic plates and shells. In particular, special values have been proposed for some of the coefficients α_i, which will be used later in this Section. Although we are considering large displacements of an elastic surface the information obtained in [12] and [13] on the coefficients α_i remains valid for our present purpose, because A in (2.30) keeps the same form for infinitesimal deformations.

Thus, let the Cosserat plate describe (approximately) the behavior of an elastic plate of thickness h, with Young's modulus E and Poisson's ratio ν, and we assume that the middle plane of the plate is inextensible.† Nondimensional coefficients β_i are introduced as follows

$$\begin{aligned} \alpha_3 &= Eh\beta_3, \qquad \alpha_4 = Eh\beta_4, \qquad \alpha_5 = Eh^3\beta_5, \\ \alpha_6 &= Eh^3\beta_6, \qquad \alpha_7 = Eh^3\beta_7, \qquad \alpha_8 = Eh^3\beta_8. \end{aligned} \tag{4.16}$$

After replacing the coefficients α_i in (4.15) through their value given in (4.16), we find that the general solution of the system is given by

$$\begin{aligned} d_1 &= \Delta_1 \operatorname{sh} \lambda_1 \frac{y}{h} + \frac{D_{21}(\lambda_2)}{D_{22}(\lambda_2)}\Delta_2 \operatorname{sh} \lambda_2 \frac{y}{h} + \frac{D_{31}(\lambda_3)}{D_{33}(\lambda_3)}\Delta_3 \operatorname{sh} \lambda_3 \frac{y}{h}, \\ d_2 &= \frac{D_{12}(\lambda_1)}{D_{11}(\lambda_1)}\Delta_1 \operatorname{sh} \lambda_1 \frac{y}{h} + \Delta_2 \operatorname{sh} \lambda_2 \frac{y}{h} + \frac{D_{32}(\lambda_3)}{D_{33}(\lambda_3)}\Delta_3 \operatorname{sh} \lambda_3 \frac{y}{h}, \\ d_3 &= d_3^0 - \frac{D_{13}(\lambda_1)}{D_{11}(\lambda_1)}\Delta_1 \operatorname{ch} \lambda_1 \frac{y}{h} - \frac{D_{23}(\lambda_2)}{D_{22}(\lambda_2)}\Delta_2 \operatorname{ch} \lambda_2 \frac{y}{h} - \Delta_3 \operatorname{ch} \lambda_3 \frac{y}{h}, \end{aligned} \tag{4.17}$$

where the λ_i's are the solution of

$$D(\lambda) = \begin{vmatrix} \beta_6\lambda^2 - (\beta_3 + \varepsilon^2\beta_8 \cos^2\alpha) & & \\ \varepsilon^2\beta_8 \sin\alpha\cos\alpha & (\beta_5+\beta_6+\beta_7)\lambda^2 - (\beta_3+\varepsilon^2\beta_8\sin^2\alpha) & \\ \varepsilon(\beta_6+\beta_7+\beta_8)\lambda\sin\alpha\cos\alpha & -\varepsilon[\beta_5+(\beta_6+\beta_7+\beta_8)\sin^2\alpha]\lambda & -\beta_8\lambda^2+[\beta_4+\varepsilon^2(\beta_5+\beta_6+\beta_7)] \end{vmatrix} = 0. \tag{4.18}$$

The parameter ε in (4.18) stands for the ratio h/R of the thickness to the radius of curvature of the deformed surface, and $D_{ij}(\lambda)$ is the cofactor of the element (ij) in the symmetric determinant $D(\lambda)$. The constants Δ_i have to be determined through the boundary conditions, and d_3^0, together with $d_1^0 = d_2^0 = 0$, is the particular solution of the system (4.15),

$$d_3^0 = \frac{\beta_4}{\beta_4 + \varepsilon^2(\beta_5+\beta_6+\beta_7)}. \tag{4.19}$$

The moment boundary conditions along the edges $y = \pm a$ are determined by $(2.27)_2$, and we have

$$M_{21}(\pm a) = 0, \qquad M_{22}(\pm a) = 0, \qquad M_{23}(\pm a) = 0. \tag{4.20}$$

The constants $\Delta_1, \Delta_2, \Delta_3$ can then be evaluated by expressing M_{21}, M_{22}, M_{23} as a function

An examination of (4.18) reveals that, for a fixed angle of pitch α, the roots λ_i are approximated by

$$\lambda_1^0 = \left(\frac{\beta_3}{\beta_6}\right)^{\frac{1}{2}}, \qquad \lambda_2^0 = \left(\frac{\beta_3}{\beta_5+\beta_6+\beta_7}\right)^{\frac{1}{2}}, \qquad \lambda_3^0 = \left(\frac{\beta_4}{\beta_8}\right)^{\frac{1}{2}}. \tag{4.21}$$

† For example, we might think of a double grid of inextensible cords (with fixed angles), between two layers of elastic material.

and the magnitude of the error is of the order ε^2. The cofactors $D_{12}(\lambda_i)$, $D_{22}(\lambda_i)$, $D_{31}(\lambda_i)$ which appear in (4.17) are all factors of ε^2; an analysis of the boundary conditions (4.20) shows that when the ratio h/a is also much smaller than 1, one has, near the edges

$$
\begin{aligned}
d_1 &= \varepsilon \frac{\beta_6+\beta_7}{(\beta_3\beta_6)^{\frac{1}{2}}} \frac{1}{2} \sin 2\alpha \, e^{-[\lambda_1^0(a-|y|)]/h} + 0(\varepsilon^2),\\
d_2 &= -\varepsilon \frac{\beta_5+(\beta_6+\beta_7)\sin^2\alpha}{\beta_3^{\frac{1}{2}}(\beta_5+\beta_6+\beta_7)^{\frac{1}{2}}} e^{-[\lambda_2^0(a-|y|)]/h} + 0(\varepsilon^2),\\
d_3 - d_3^0 &= 0(\varepsilon^2).
\end{aligned} \tag{4.22}
$$

By pursuing the analogy between a Cosserat plate and its elastic plate counterpart, Green and Naghdi [13] have identified as follows the coefficients β_i in (4.16),

$$
\begin{aligned}
\beta_3 &= \frac{5}{12(1+\nu)}, \qquad \beta_4 = \frac{(1-\nu)}{(1+\nu)(1-2\nu)},\\
\beta_5 &= \frac{\nu}{12(1-\nu^2)}, \qquad \beta_6 = \beta_7 = \frac{1}{24(1+\nu)},
\end{aligned} \tag{4.23}
$$

where the value of β_3 is based on a comparison with the problem of torsion of a rectangular strip.

Equations (4.21)–(4.23) show that the director components d_1 and d_2 differ from zero on a layer concentrated along the edges, with a width of the order h. The boundary layer corresponding to d_2 has the same origin as the phenomenon discussed by Fung and Wittrick [8], where however the layer has a width of order $\sqrt{(hR)}$; the difference arises because of the true inextensibility of the middle surface in the present problem.

Finally, the component (d_3-1) differs from zero in a layer along the edge, the width of which depends on the value of β_8. If β_8 tends to zero, $(4.21)_3$ shows that the width of the boundary layer increases, but the analysis reveals that Δ_3 decreases simultaneously; the physical meaning of Δ_3 corresponds to a normal strain effect along the edges.

In view of the above comments, and the relations $(2.4)_2$, (2.31) and (4.17), it appears that the nonvanishing components of the curve force vector are concentrated in the boundary layers. This last remark provides further significance to the subject of inextensible surfaces.

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Абстракт—Выводится система зависимостей поля и конститутивных зависимостей, вместе с граничными условиями, для конечных деформаций нерастяжимых поверхностей Коссера. Этая теория применяется к изгибу прямоугольной пластинки, преобразовываемой в замкнутый круглый цилиндр конечной длины и к деформации длинной прямоугольной пластинке, преобразовываемой в спиральную полосу—(подобную столбу, окрашенного красным и белым цветом по спирали, служащему вывеской парикмахера в С Ш А).